Solutions to the 2017 Olympiad Cayley Paper

C1. Four times the average of two different positive numbers is equal to three times the greater one. The difference between the numbers is three less than the average.

What are the two numbers?

Solution

It is good to give the numbers names, so let them be a and b, where b is the larger. The first fact we are given says that

$$4 \times \frac{a + b}{2} = 3b,$$

and the second says that

$$b - a = \frac{a + b}{2} - 3.$$

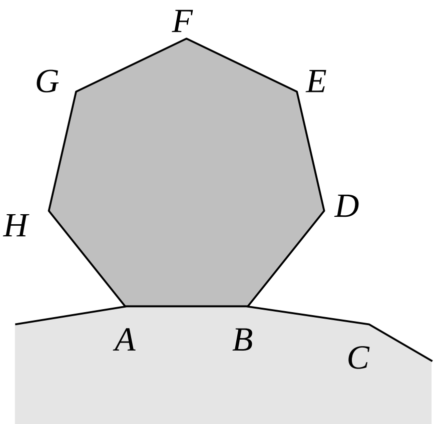
We may simplify the first fact to obtain $2(a + b) = 3b$, that is, $2a = b$; multiplying the second fact by two, we get $2b - 2a = a + b - 6$, that is, $b = 3a - 6$.

Equating these values of b, we get $2a = 3a - 6$, that is, $6 = a$, and hence $b = 12$.

So the two numbers are 6 and 12.

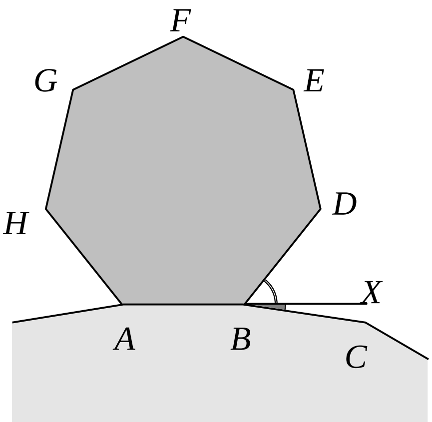
C2. The diagram shows three adjacent vertices A, B and C of a regular polygon with forty-two sides, and a regular heptagon $ABDEFGH$. The polygons are placed together edge-to-edge.

Prove that triangle BCD is equilateral.



Solution

One can prove that a triangle is equilateral in several different ways, and the diagram gives a few hints about which will be easiest. Two of the sides and one of the angles look more important than the other side and the other two angles, so we'll talk about them.



All sides of the heptagon are of equal length, so $AB = BD$; similarly all sides of the 42-sided polygon are of equal length, so $AB = BC$. Thus $BD = BC$, making triangle DBC isosceles.

We'll now compute the angle DBC by extending the common side of the two polygons to X, as shown in the diagram alongside. We see that angle DBC is the sum of the external angles of two regular polygons.

Thus

$$\begin{aligned}
\angle DBC &= \angle DBX + \angle XBC \\
&= \frac{360^\circ}{7} + \frac{360^\circ}{42} \\
&= \left(\frac{1}{7} + \frac{1}{42}\right) \times 360^\circ \\
&= \frac{1}{6} \times 360^\circ \\
&= 60^\circ.
\end{aligned}$$

Hence DBC is an isosceles triangle with an angle of 60°, which means that it is equilateral.

C3. Peaches spends exactly £3.92 on some fruit, choosing from apples costing 20p each and pears costing 28p each.

How many of each type of fruit might she have bought?

Solution

First we should try to translate this word problem into algebra. Let a be the number of apples and p the number of pears, then what we have is that $20a + 28p = 392$.

All these numbers are divisible by 4, so we may divide each term by 4 to obtain $5a + 7p = 98$.

Now 98 is a multiple of 7, and $7p$ is always a multiple of 7, so $5a$ is also a multiple of 7. This means that a itself is a multiple of 7.

So we can have $a = 0$ (giving $p = 14$), or $a = 7$ (giving $p = 9$), or $a = 14$ (giving $p = 4$). We can't take a to be a higher multiple of 7, such as 21 or more, because then $5a \geqslant 105$, so that $20a \geqslant 240$, which means that Peaches would be spending more on apples than she spends altogether.

Hence the numbers of each type of fruit that Peaches might have bought are shown in the following table.

Apples	Pears
0	14
7	9
14	4

C4. The point X lies inside the square $ABCD$ and the point Y lies outside the square, in such a way that triangles XAB and YAD are both equilateral.

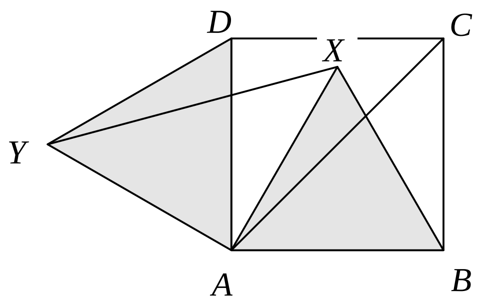
Prove that $XY = AC$.

Solution

See the diagram alongside.

We'll start by working out some angles at A, since that is the vertex where the most is going on.

Firstly, $\angle DAB = 90°$ since $ABCD$ is a square. Also, each of angles XAB and YAD is $60°$ since each of the triangles XAB and YAD is equilateral.



Therefore

$$\begin{aligned}\angle YAX &= \angle YAD + \angle DAB - \angle XAB \\ &= 60° + 90° - 60° \\ &= 90°.\end{aligned}$$

Also, since each of the equilateral triangles shares a side with the square, they have sides that are the same length as the side of the square. In particular, $YA = AB$ and $AX = BC$.

It follows that the triangles YAX and ABC are congruent (SAS), and hence $AC = XY$.

C5. In a sports league there are four teams and every team plays every other team once. A team scores 3 points for a win, 1 point for a draw, and 0 points for a loss.

What is the smallest number of points that a team could have at the end of the league and still score more points than each of the other teams?

Solution

There are six games, and each game contributes 2 or 3 points in total, depending on whether it was drawn or won, so the total number of points in the league is $12 + w$, where w is the number of won games. So the total number of points in the league is at least 12 and at most 18.

Consider the number of points at the end of the league scored by a team scoring more points than every other team.

3 points or fewer

There are at least 9 other points scored in the league, and so it's not possible for every other team to score 2 points or fewer: that makes at most 6 points.

4 points

The only way for a team to score 4 points is $3 + 1 + 0$, so there are at least two won games. That means that the total number of points is at least 14 points, so the other teams have at least 10 points between them. It is thus not possible for each of them to score 3 points or fewer: that makes at most 9 points.

5 points

This is possible: label the teams A, B, C and D; if A beats D but every other game is drawn, then A has 5 points, two of the other teams have 3 points and one has 2 points.

Hence the answer is 5.

C6. We write ‘pq’ to denote the two-digit integer with tens digit p and units digit q.

For which values of a, b and c are the two fractions $\dfrac{‘ab’}{‘ba’}$ and $\dfrac{‘bc’}{‘cb’}$ equal and different from 1?

Solution

Write this using ‘proper’ algebra: the question asks us to find solutions to

$$\frac{10a + b}{10b + a} = \frac{10b + c}{10c + b}, \tag{1}$$

where a, b and c are integers from 1 to 9 (none of them can be 0 because ‘ab’ and so on are two-digit numbers).

Multiplying each side of (1) by $(10b + a)(10c + b)$, we obtain

$$(10a + b)(10c + b) = (10b + c)(10b + a),$$

that is,

$$100ac + 10(ab + bc) + b^2 = 100b^2 + 10(ab + bc) + ac,$$

so that

$$100ac + b^2 = 100b^2 + ac.$$

But a, b and c are integers between 1 and 9, hence b^2 and ac are between 1 and 81. Therefore b^2 and ac are less than 100, whereas $100ac$ and $100b^2$ are greater than 100. It follows that $b^2 = ac$.

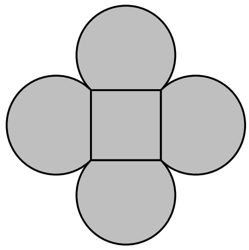
Note that $a \neq b$ (otherwise $a = b$ and then $b = c$ too, so that $a = b = c$).

Remembering that a, b and c are integers from 1 to 9, we get the values shown in the following table.

b	a, c in either order
2	1, 4
3	1, 9
4	2, 8
6	4, 9

Solutions to the 2017 Olympiad Hamilton Paper

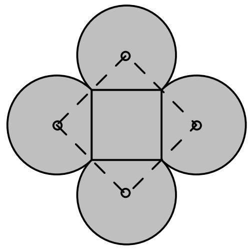
H1. The diagram shows four equal arcs placed on the sides of a square. Each arc is a major arc of a circle with radius 1 cm, and each side of the square has length $\sqrt{2}$ cm.



What is the area of the shaded region?

Solution

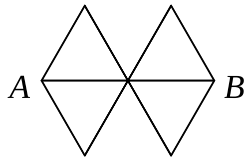
Join the centres of the circles, as shown in the diagram alongside, to form a quadrilateral whose sides have length 1 cm + 1 cm. Each angle of this quadrilateral is equal to 90° from the converse of Pythagoras' Theorem, since we are given the 'inner' square has sides of length $\sqrt{2}$ cm. Because it has equal sides. it follows that the quadrilateral is a square.



The shaded region comprises this square and four sectors of circles, each of radius 1 cm and angle 270°, thus its area is equal to $2 \times 2 + 4 \times \frac{3}{4} \times \pi \times 1^2$, in cm^2.

Therefore, in cm^2, the shaded area is equal to $4 + 3\pi$.

H2. A ladybird walks from A to B along the edges of the network shown. She never walks along the same edge twice. However, she may pass through the same point more than once, though she stops the first time she reaches B.

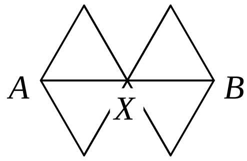


How many different routes can she take?

Solution

Label the centre point X, as shown in the diagram alongside.

Clearly any route that the ladybird takes from A to B passes through X, and she stops the first time she reaches B.



Therefore the number of different routes that the ladybird can take is equal to

(the number of routes from A to X) × (the number from X to B).

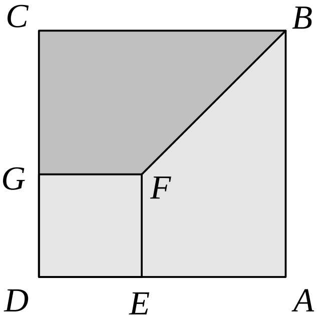
The number of routes from A to X is equal to

(the number of 'direct' routes from A to X) + (the number from A to X that visit X twice),

which is $3 + 3 \times 2$.

However, the number of routes from X to B is just 3, since the ladybird stops the first time she reaches B, so that it is not possible for her to visit X again. Thus the total number of different routes that the ladybird can take is 9×3, which equals 27.

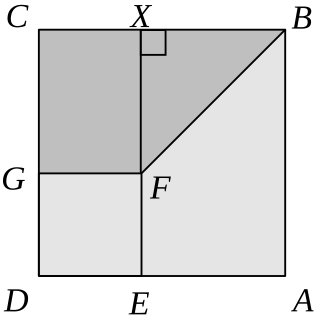
H3. The diagram shows squares *ABCD* and *EFGD*. The length of *BF* is 10 cm. The area of trapezium *BCGF* is 35 cm^2.



What is the length of *AB*?

Solution

The point *F* lies on the diagonal *BD* of the square *ABCD*, so that $\angle FBC$ is equal to 45°. Let point *X* lie on *BC* so that $\angle FXB = 90°$, as shown in the diagram alongside. Then $\angle XFB = 45°$ from the angle sum of triangle *BXF*; it follows from 'sides opposite equal angles are equal' that $BX = XF$.



Now, using Pythagoras' Theorem in triangle *BXF*, we obtain $XF = 5\sqrt{2}$ cm. But *CGFX* is a rectangle, so that $CG = XF$.

Let the length of *AB* be a cm. Then

$\frac{1}{2} \times 5\sqrt{2} \times \left(2a - 5\sqrt{2}\right) = 35$ and so $2a - 5\sqrt{2} = \dfrac{14}{\sqrt{2}}$.

Hence $2a = 5\sqrt{2} + 14\dfrac{\sqrt{2}}{2}$, so that $a = 6\sqrt{2}$.

Therefore the length of *AB* is $6\sqrt{2}$ cm.

H4. The largest of four different real numbers is d. When the numbers are summed in pairs, the four largest sums are 9, 10, 12 and 13.

What are the possible values of d?

Solution

Let the other three different numbers be a, b and c, in increasing order. Then each of them is less than d, so that $c + d$ is the largest sum of a pair. The next largest is $b + d$, because it is larger than any other sum of a pair. But we do not know whether $b + c$ or $a + d$ is next (though each of these is larger than $a + c$, which in turn is larger than $a + b$). There are thus two cases to deal with, depending on whether $b + c \leqslant a + d$ or $a + d < b + c$.

$\boldsymbol{b + c \leqslant a + d}$

We have

$$b + c = 9, \tag{1}$$

$$a + d = 10, \tag{2}$$

$$b + d = 12 \tag{3}$$

and

$$c + d = 13. \tag{4}$$

From equations (1), (3) and (4), we find that $2d = 16$, so that $d = 8$.

$\boldsymbol{a + d < b + c}$

We have

$$a + d = 9, \tag{5}$$

$$b + c = 10, \tag{6}$$

$$b + d = 12 \tag{7}$$

and

$$c + d = 13. \tag{8}$$

From equations (6) to (8), we find that $2d = 15$, so that $d = 7.5$.

In each case, it is possible to find the values of a, b and c from the equations, and to check that these fit the conditions in the question.

Therefore the possible values of d are 7.5 and 8.

H5. In the trapezium $ABCD$, the lines AB and DC are parallel, $BC = AD$, $DC = 2AD$ and $AB = 3AD$.

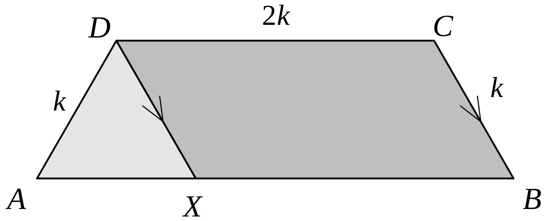
The angle bisectors of $\angle DAB$ and $\angle CBA$ intersect at the point E.

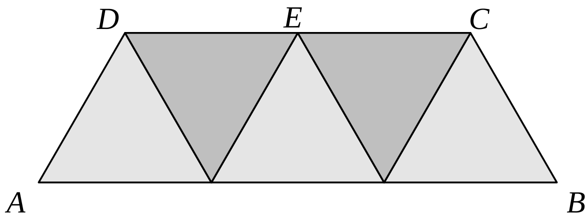
What fraction of the area of the trapezium $ABCD$ is the area of the triangle ABE?

Solution

Let $BC = AD = k$, so that $DC = 2k$ and $AB = 3k$, and let the point X lie on AB so that $XBCD$ is a parallelogram, as shown in the diagram on the left below. It follows that $DX = k$ and $XB = 2k$ (opposite sides of a parallelogram), so that $AX = k$.

Hence triangle AXD has three equal sides—it is therefore an equilateral triangle. In particular, this means that angle DAX is equal to 60°.





As a consequence, the trapezium $ABCD$ is actually made up from five equilateral triangles, as shown in the diagram on the right above.

Now the triangle ABE comprises one equilateral triangle and two half-rhombuses. The area of the two half-rhombuses is equal to the area of two equilateral triangles.

Therefore the area of the triangle ABE is $\frac{3}{5}$ of the area of the trapezium $ABCD$.

H6. Solve the pair of simultaneous equations

$$x^2 + 3y = 10 \qquad \text{and}$$

$$3 + y = \frac{10}{x}.$$

Solution

First, let us number the two given equations, so that it is easy to refer to them.

$$x^2 + 3y = 10 \tag{1}$$

$$3 + y = \frac{10}{x} \tag{2}$$

It is possible to eliminate one of the two unknowns by substituting from equation (2) into equation (1), but this leads to a cubic equation. We present another method that avoids this.

By subtracting $x \times$ equation (2) from equation (1), we get

$$x^2 + 3y - 3x - xy = 0$$

so that

$$(x - 3)(x - y) = 0.$$

Hence either $x = 3$ or $x = y$. We deal with each of these two cases separately.

$\boldsymbol{x = 3}$

Using equation (1), say, we obtain $y = \frac{1}{3}$.

$\boldsymbol{x = y}$

Using equation (1) we obtain

$$x^2 + 3x = 10$$

so that

$$x^2 + 3x - 10 = 0.$$

Hence

$$(x - 2)(x + 5) = 0,$$

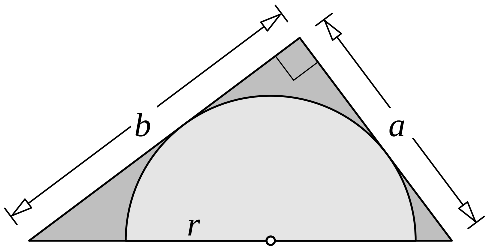
and therefore either $x = 2$ or $x = -5$. When $x = 2$ then $y = 2$; when $x = -5$ then $y = -5$.

By checking in the two given equations, we find that all three solutions are valid. Thus there are three solutions of the simultaneous equations, namely

$$\text{either } x = -5 \text{ and } y = -5,$$
$$\text{or } x = 2 \text{ and } y = 2,$$
$$\text{or } x = 3 \text{ and } y = \tfrac{1}{3}.$$

Solutions to the 2017 Olympiad Maclaurin Paper

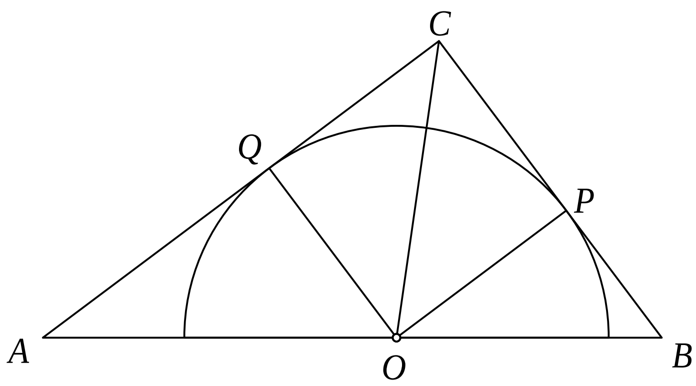
M1. The diagram shows a semicircle of radius r inside a right-angled triangle. The shorter edges of the triangle are tangents to the semicircle, and have lengths a and b. The diameter of the semicircle lies on the hypotenuse of the triangle.



Prove that

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}.$$

Solution



Label the points as shown above, and join O to P, Q and C.

Method 1

The area of triangle AOC is $\frac{1}{2}rb$, that of triangle BOC is $\frac{1}{2}ra$ and that of triangle ABC is $\frac{1}{2}ab$.

Hence, adding, we have

$$\tfrac{1}{2}ab = \tfrac{1}{2}rb + \tfrac{1}{2}ra.$$

Dividing each term by $\frac{1}{2}abr$, we obtain

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}.$$

Method 2

$AQ = b - r$. The triangles AOQ and ABC are similar (AA), so that

$$\frac{r}{b - r} = \frac{a}{b}.$$

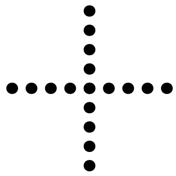
Multiplying each side by $b\,(b - r)$, we get

$$rb = ab - ar$$

and now, dividing each term by abr, we obtain

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}.$$

M2. How many triangles (with non-zero area) are there with each of the three vertices at one of the dots in the diagram?



Solution

Method 1

There are 17 dots in the array, and we must choose 3 of them to obtain a triangle. This can be done in $\binom{17}{3} = 680$ ways. However, some of these triangles have zero area.

The triangle will have zero area if we choose all three dots on the same line. Hence the number of triangles of zero area is $2 \times \binom{9}{3} = 2 \times 84 = 168$.

So there are 512 triangles of non-zero area.

Method 2

We may choose two points on the line across the page in $\binom{9}{2} = 36$ ways, and one point on the line up the page in 8 ways. These choices give rise to $8 \times 36 = 288$ triangles of non-zero area.

Similarly we obtain another 288 triangles from two points on the line up the page and one on the line across the page.

But we have counted twice triangles with a vertex at the point where the lines meet, and there are $8 \times 8 = 64$ of these. So altogether we have $2 \times 288 - 64 = 512$ triangles.

M3. How many solutions are there to the equation

$$m^4 + 8n^2 + 425 = n^4 + 42m^2,$$

where m and n are integers?

Solution

By 'completing the square', we may rewrite the equation in the form

$$\left(m^2 - 21\right)^2 = \left(n^2 - 4\right)^2.$$

Then, taking the square root of each side, we get

$$m^2 - 21 = \pm\left(n^2 - 4\right).$$

Hence there are two cases to consider.

$\boldsymbol{m^2 - 21 = n^2 - 4}$

In this case, we have

$$m^2 - n^2 = 21 - 4,$$

so that

$$(m - n)(m + n) = 17.$$

Therefore, because 17 is prime, $m - n$ and $m + n$ are equal to 1 and 17, or -1 and -17, in some order. Thus in this case there are four solutions for (m, n), namely $\left(\pm 9, \pm 8\right)$.

$\boldsymbol{m^2 - 21 = -\left(n^2 - 4\right)}$

In this case, we have

$$m^2 + n^2 = 21 + 4.$$

Hence

$$m^2 + n^2 = 5^2$$

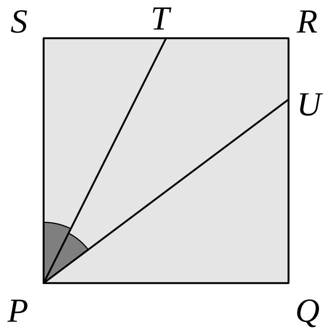
Now a square is non-negative, so that $-5 \leqslant m, n \leqslant 5$.

Thus in this case there are twelve solutions for (m, n), namely $(0, \pm 5)$, $(\pm 5, 0)$, $(\pm 3, \pm 4)$ and $(\pm 4, \pm 3)$.

Therefore altogether there are sixteen solutions to the given equation.

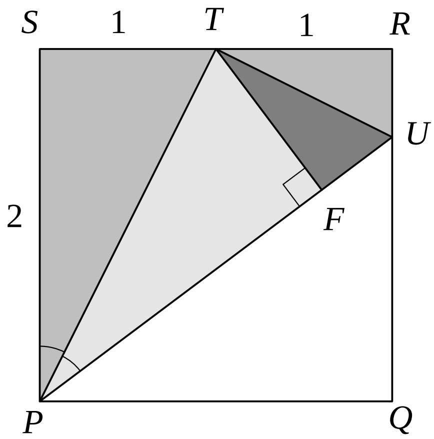
M4. The diagram shows a square $PQRS$ with sides of length 2. The point T is the midpoint of RS, and U lies on QR so that $\angle SPT = \angle TPU$.

What is the length of UR?



Solution

Let F be the point on PU so that $\angle TFP = 90°$ and join T to F and U, as shown. Then triangles PTS and PTF are congruent (AAS), so that $TF = 1$.



Hence triangles TUR and TUF are congruent (RHS), so that $\angle RTU = \angle UTF$.

Now the four angles at T are angles on the straight line RTS, so they add up to 180°. It follows that $\angle RTU = \angle SPT$.

Therefore triangles RTU and SPT are similar (AA), so that

$$\frac{UR}{RT} = \frac{TS}{SP} = \frac{1}{2}.$$

Thus $UR = \frac{1}{2}$.

M5. Solve the pair of simultaneous equations

$$(a + b)(a^2 - b^2) = 4 \qquad \text{and}$$

$$(a - b)(a^2 + b^2) = \frac{5}{2}.$$

Solution

Since $a^2 - b^2 = (a - b)(a + b)$, we may write the first equation as $(a + b)^2(a - b) = 4$.

Note that $a - b \neq 0$, since this would make the left-hand side of both equations zero, which is impossible. Hence we can divide the first equation by the second and cancel the term $a - b$ to produce

$$\frac{(a + b)^2}{a^2 + b^2} = \frac{8}{5}.$$

Multiplying each side by $5(a^2 + b^2)$ we get

$$5(a + b)^2 = 8(a^2 + b^2).$$

When we multiply this out and collect like terms, we obtain

$$\begin{aligned} 0 &= 3a^2 - 10ab + 3b^2 \\ &= (3a - b)(a - 3b), \end{aligned}$$

so either $a = 3b$ or $b = 3a$.

We substitute each of these in turn back into the first equation.

$\boldsymbol{a = 3b}$

Then $4b \times 8b^2 = 4$, so that $b = \frac{1}{2}$ and $a = \frac{3}{2}$.

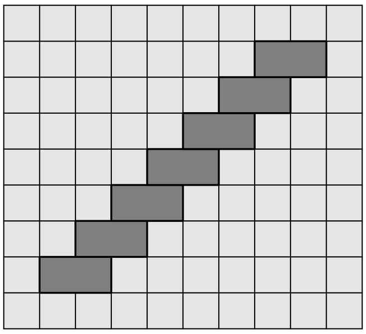
$\boldsymbol{b = 3a}$

Then $4a \times (-8a^2) = 4$, so that $a = -\frac{1}{2}$ and $b = -\frac{3}{2}$.

Hence we have two solutions $(a, b) = \left(\frac{3}{2}, \frac{1}{2}\right)$ or $(a, b) = \left(-\frac{1}{2}, -\frac{3}{2}\right)$. These solutions should be checked by substituting back into the second equation.

M6. The diagram shows a 10 × 9 board with seven 2 × 1 tiles already in place.

What is the largest number of additional 2 × 1 tiles that can be placed on the board, so that each tile covers exactly two 1 × 1 cells of the board, and no tiles overlap?

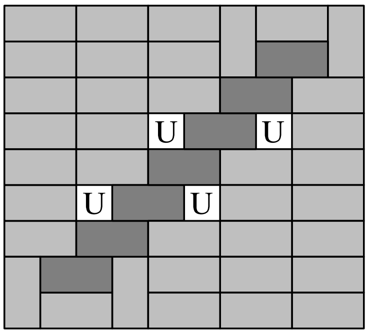


Solution

The first observation is that it is possible to add a further 36 tiles to the grid. The diagram shows one way of doing this: the additional tiles are lighter grey and the uncovered squares are indicated.

There are 36 additional tiles in the grid, and four squares are left uncovered.

We show that you cannot improve on this.



Colour the grid like a chessboard with alternating grey and white cells, as shown in Figure 1.

Notice that any tile will cover one cell of each colour.

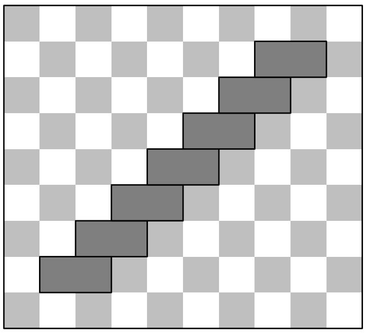


Figure 1

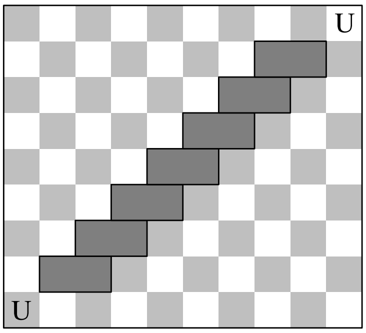


Figure 2

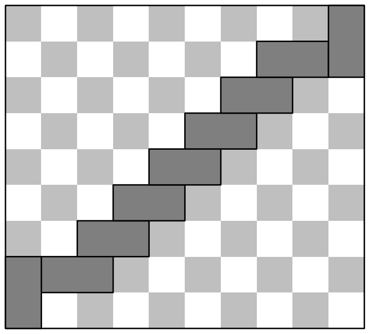


Figure 3

Suppose that each corner is left uncovered, as shown in Figure 2. Then the remainder of the board consists of two separate ‘staircases’.

The upper staircase has 17 grey and 20 white cells, so that at most seventeen 2 × 1 tiles may be placed here. Similarly for the lower staircase: at most seventeen 2 × 1 tiles may be placed there. In other words, with this arrangement, at most 34 additional tiles may be placed.

The only way to cover the corners whilst also reducing the excess in both staircases is to place tiles in the corners as shown in Figure 3. This reduces the number of white cells in the upper staircase by one, and reduces the number of grey cells in the lower staircase by one. Once again at most seventeen tiles may be placed in each staircase, achieving at most 36 additional tiles in total.

Therefore the greatest number of additional 2 × 1 tiles that can be placed on the board is 36.